

Hoffman's coclique bound for normal regular digraphs, and nonsymmetric association schemes.

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Abstract

We extend Hoffman's coclique bound for regular digraphs with the property that its adjacency matrix is normal, and discuss cocliques attaining the inequality. As a consequence, we characterize skew-Bush-type Hadamard matrices in terms of digraphs. We present some normal digraphs whose vertex set is decomposed into disjoint cocliques attaining the bound. The digraphs provided here are relation graphs of some nonsymmetric association schemes.

1 Introduction

Spectral graph theory for undirected graph has been studied very well [2]. Using the eigenvalues of the adjacency matrix of a graph, we obtain several inequalities for parameters of the graph, such as the clique number, the independence number, the chromatic number, etc. Hoffman gave an upper bound for independence number of regular graphs to use the eigenvalues of the adjacency matrix.

In this paper we extend the Hoffman bound for normal regular digraphs. Here, a normal digraph means a digraph with the adjacency matrix being normal, see Section 2.1. A coclique attaining the upper bound is also studied. In Section 4, we use the bound to characterize skew-Bush-type (or skew-checked) Hadamard matrices in terms of doubly regular asymmetric digraphs with some properties. This result is an analogy of the result by Wallis [12] that there exists a symmetric Bush-type Hadamard matrix of order $4n^2$ if and only if a strongly regular graph with parameters $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$ such that the vertex set is decomposed into $2n$ disjoint cocliques of size $2n$. Note that the cocliques of size $2n$ in the strongly regular graph attain Hoffman's bound.

A coclique attaining Hoffman's coclique bound in a strongly regular graph Γ is a clique attaining the clique bound in the complement of Γ . A spread of

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a strongly regular graph is a set of disjoint cliques attaining the clique bound. In [1, 3, 5], a spread of strongly regular graphs is extensively studied. In Section 5, 6 we provide some normal digraphs with the vertex set decomposed into disjoint cocliques attaining the upper bound. All of them are relation graphs of association schemes.

2 Preliminaries

2.1 Digraphs

A *digraph* Γ is a pair (X, E) such that the *vertex set* X is a finite set and the *edge set* or *arc set* E is a subset of $X \times X$ with $E \cap \{(x, x) \mid x \in X\} = \emptyset$. The *adjacency matrix* of Γ is a $(0, 1)$ -matrix with rows and columns indexed by the elements of X such that $A_{xy} = 1$ if $(x, y) \in E$ and $A_{xy} = 0$ otherwise. A digraph Γ is *asymmetric* if $(x, y) \in E$ implies $(y, x) \notin E$, namely $A + A^T$ is a $(0, 1)$ -matrix, where A^T denotes the transpose of A . A digraph Γ is *normal* if the adjacency matrix A is normal, namely $AA^T = A^T A$ holds. A digraph Γ is *k-regular* if $|\{y \in X \mid (x, y) \in E\}| = |\{y \in X \mid (y, x) \in E\}| = k$ for any vertex x .

A digraph Γ is *normally regular with parameters* (n, k, λ, μ) if Γ is asymmetric, the number of vertices of Γ is n and the adjacency matrix A of Γ satisfies

$$AA^T = kI_n + \lambda(A + A^T) + \mu(J_n - I_n - A - A^T), \quad (1)$$

where I_n is the identity matrix of order n and J_n is the all ones matrix of order n . It was shown in [8] that a normally regular digraph is indeed normal. A *doubly regular asymmetric digraph* Γ with parameters (v, k, λ) is a normally regular digraph with parameters (v, k, λ, λ) .

A subset C in X is a *coclique* (or an *independence set*) in Γ if $(x, y) \notin E$ for any $x, y \in C$.

A digraph Γ is *strongly connected* if for any distinct vertices x, y , there exist vertices x_0, \dots, x_s such that $x_0 = x$, $x_s = y$ and $(x_i, x_{i+1}) \in E$ for any $i \in \{0, 1, \dots, s-1\}$.

The following lemma will be used in Proposition 4.1.

Lemma 2.1. *Let Γ be a normally regular digraph with parameters (n, k, λ, μ) with adjacency matrix A . Assume that $A + A^T = J_n - I_r \otimes J_{n/r}$ for some positive integer r dividing n . Then the eigenvalues of A are k , $\pm\sqrt{-k + \mu}$, or $-n/(2r) \pm \sqrt{k - \mu + (-\lambda + \mu)n/r - n^2/(4r^2)}$.*

Proof. The valency k is an eigenvalue of A with the all-ones vector as an eigenvector. Let α be an eigenvalue whose eigenvector is orthogonal to the all-ones vector. By the equation (1), we have

$$\alpha\bar{\alpha} = k + \lambda(\alpha + \bar{\alpha}) + \mu(-1 - \alpha - \bar{\alpha}). \quad (2)$$

Since $A + A^T = J_n - I_r \otimes J_{n/r}$, the real part of α is $-n/(2r)$ or 0. By (2), α is the desired value. \square

2.2 Association schemes

A *commutative association scheme of class d* with vertex set X of size n is a set of non-zero $(0, 1)$ -matrices A_0, \dots, A_d , which are called *adjacency matrices*, with rows and columns indexed by X , such that:

- (i) $A_0 = I_n$.
- (ii) $\sum_{i=0}^d A_i = J_n$.
- (iii) For any $i \in \{0, 1, \dots, d\}$, $A_i^T \in \{A_0, A_1, \dots, A_d\}$.
- (iv) For any $i, j \in \{0, 1, \dots, d\}$, $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$ for some p_{ij}^k 's.
- (v) For any $i, j \in \{0, 1, \dots, d\}$, $A_i A_j = A_j A_i$.

The association scheme is said to be *symmetric* if all A_i are symmetric, *non-symmetric* otherwise. The *intersection matrix* B_i ($i \in \{0, 1, \dots, d\}$) is $B_i = (p_{ij}^k)_{j,k=0}^d$.

A digraph $\Gamma = (X, E)$ is a *relation graph* of an association scheme with vertex set X if the adjacency matrix of Γ is one of the adjacency matrices of the association scheme.

The vector space spanned by A_i 's forms a commutative algebra, denoted by \mathcal{A} and called the *Bose-Mesner algebra* or *adjacency algebra*. There exists a basis of \mathcal{A} consisting of primitive idempotents, say $E_0 = (1/n)J_n, E_1, \dots, E_d$. Since $\{A_0, A_1, \dots, A_d\}$ and $\{E_0, E_1, \dots, E_d\}$ are two bases of \mathcal{A} , there exist the change-of-bases matrices $P = (P_{ij})_{i,j=0}^d$, $Q = (Q_{ij})_{i,j=0}^d$ so that

$$A_j = \sum_{i=0}^d P_{ij} E_i, \quad E_j = \frac{1}{n} \sum_{i=0}^d Q_{ij} A_i.$$

The matrix P (Q respectively) is said to be the *first* (*second respectively*) *eigenmatrix*.

3 Hoffman's bound for normal digraphs

In this section, we give an upper bound for the size of cocliques in a normal digraph in terms of eigenvalues of the adjacency matrix of the digraph Γ . The upper bound is referred to as the *Hoffman bound*. For a digraph with adjacency matrix A , define $\theta_{\min} = \min\{\operatorname{Re}(\theta) \mid \theta \text{ is an eigenvalue of } A\}$, $\operatorname{Re}(\theta)$ is the real part of θ . Note that for a normal graph, θ_{\min} is negative since the trace of A is zero unless Γ has no edge.

Proposition 3.1. *Let n, k be positive integers. Let $\Gamma = (X, E)$ be a strongly connected k -regular normal digraph with n vertices and adjacency matrix A . For a coclique C in Γ , it holds that*

$$|C| \leq \frac{n(-\theta_{\min})}{k - \theta_{\min}}. \quad (3)$$

Moreover the following hold.

- (i) If equality holds in (3), then $|\{y \in C \mid (x, y) \in E\}| + |\{y \in C \mid (y, x) \in E\}| = -2\theta_{\min}$ for any $x \in X \setminus C$.
- (ii) If equality holds in (3) and the number of eigenvalues with real part equal to θ_{\min} is exactly one, then $|\{y \in C \mid (x, y) \in E\}| = -\theta_{\min}$ for any $x \in X \setminus C$.

Proof. Let $\theta_1, \dots, \theta_{l+2m}$ be the eigenvalues of A such that $\theta_i \in \mathbb{R}$ for any $i \in \{1, \dots, l\}$ and $\overline{\theta_{l+j}} = \theta_{l+m+j} \notin \mathbb{R}$ for any $j \in \{1, \dots, m\}$. Let E_i be the orthogonal projection onto the eigenspace of θ_i . Then $\overline{E_{l+j}} = E_{l+m+j}$ for any $j \in \{1, \dots, m\}$. Since k is an eigenvalue, we set $\theta_1 = k$. Since Γ is strongly connected, $E_1 = \frac{1}{n}J_n$.

Let χ be the characteristic column vector of C . Since C is a coclique of Γ , it holds that

$$\chi^T A \chi = 0. \quad (4)$$

On the other hand we estimate the value $\chi^T A \chi$ to use the formula $A = \sum_{i=1}^{l+2m} \theta_i E_i$ as follows:

$$\begin{aligned} \chi^T A \chi &= \chi^T \left(\sum_{i=1}^{l+2m} \theta_i E_i \right) \chi = \sum_{i=1}^{l+2m} \theta_i \chi^T E_i \chi \\ &= k \chi^T E_1 \chi + \sum_{i=2}^l \theta_i \chi^T E_i \chi + \sum_{i=1}^{2m} \frac{\theta_{l+i} + \overline{\theta_{l+i}}}{2} \chi^T E_{l+i} \chi \\ &\geq k \chi^T E_1 \chi + \theta_{\min} \sum_{i=2}^{l+2m} \chi^T E_i \chi \\ &= k \chi^T E_1 \chi + \theta_{\min} \chi^T (I_n - E_1) \chi \\ &= \frac{(k - \theta_{\min})|C|^2}{n} + \theta_{\min}|C| \end{aligned} \quad (5)$$

Combining (4) and (5), we obtain $|C| \leq n(-\theta_{\min})/(k - \theta_{\min})$.

A coclique C meets the upper bound if and only if $\chi^T E_i \chi = 0$ for i such that $i \geq 2$ and $\text{Re}(\theta_i) \neq \theta_{\min}$.

(i): Let $A + A^T = \sum_{i=1}^t \tau_i F_i$ be the spectrum decomposition of $A + A^T$, and set $\tau_1 = 2k$ and $\tau_t = 2\theta_{\min}$. Since $F_i \chi = 0$ for $i \in \{2, 3, \dots, t-1\}$,

$$\begin{aligned} (A + A^T)\chi &= 2kF_1\chi + \tau_t F_t \chi = 2kF_1\chi + \tau_t \sum_{i=2}^t F_i \chi = 2kF_1\chi + \tau_t(I_n - F_1)\chi \\ &= \tau_t \chi + (2k - \tau_t) \frac{1}{n} J_n \chi = \tau_t \chi + (2k - \tau_t) \frac{|C|}{n} \mathbf{1} = (-2\theta_{\min})(\mathbf{1} - \chi), \end{aligned} \quad (6)$$

where $\mathbf{1}$ is the all-ones vector. The equation (6) is equivalent to the condition that the $|\{y \in C \mid (x, y) \in E\}| + |\{y \in C \mid (y, x) \in E\}| = -2\theta_{\min}$ for any $x \in X \setminus C$.

(ii): Let θ_s satisfy $\operatorname{Re}(\theta_s) = \theta_{\min}$. Then $\theta_s = \theta_{\min}$. Indeed, if $\theta_s \in \mathbb{C} \setminus \mathbb{R}$, then $\overline{\theta_s}$ also satisfies $\operatorname{Re}(\overline{\theta_s}) = \theta_{\min}$. This contradicts to the assumption. In this case,

$$\begin{aligned} A\chi &= kE_1\chi + \theta_s E_s\chi = kE_1\chi + \theta_s \sum_{i=2}^s E_i\chi = kE_1\chi + \theta_s(I_n - E_1)\chi \\ &= \theta_s\chi + (k - \theta_s)\frac{1}{n}J_n\chi = \theta_s\chi + (k - \theta_s)\frac{|C|}{n}\mathbf{1} = (-\theta_{\min})(\mathbf{1} - \chi). \end{aligned} \quad (7)$$

The equation (7) is equivalent to the condition that the size of $\{y \in C \mid (x, y) \in E\} = -\theta_{\min}$ for any $x \in X \setminus C$. \square

Remark 3.2. Assume that a normally regular digraph Γ satisfies the assumptions of Lemma 2.1. By $A + A^T = J_n - I_r \otimes J_{n/r}$, the valency k of Γ is $\frac{n(r-1)}{2r}$. Thus the right hand side of the bound in Proposition 3.1 is n/r . Then the cocliques represented as the main diagonal blocks in A attain the bound in Proposition 3.1.

4 A characterization of skew-Bush-type Hadamard matrices

A *Hadamard matrix of order n* is an $n \times n$ $(1, -1)$ -matrix such that $HH^T = nI_n$. A Hadamard matrix H of order $4n^2$ is of *Bush-type (or checkered)* if $H = (H_{ij})_{i,j=1}^{2n}$, where H_{ij} is a $2n \times 2n$ matrix for any $i, j \in \{1, \dots, 2n\}$, such that $H_{ii} = J_{2n}$ for any $i \in \{1, \dots, 2n\}$ and $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$ for any distinct $i, j \in \{1, \dots, 2n\}$. A Bush-type Hadamard matrix $H = (H_{ij})_{i,j=1}^{2n}$ of order $4n^2$ is of *skew-Bush-type (or skew-checkered)* if $H - I_{2n} \otimes J_{2n}$ is skew-symmetric.

It was shown by Haemers and Tonchev in [5] that some symmetric association scheme of class 3 exists if and only if strongly regular graphs with vertex set being decomposed into disjoint cliques attaining Hoffmann's clique bound. It was shown by Wallis in [12], that there exists a symmetric Bush-type Hadamard matrix of order $4n^2$ if and only if there exists a strongly regular graph with parameters $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$ such that the vertex set is decomposed into $2n$ disjoint cocliques of size $2n$ (see also [11, Lemma 1.1]).

Digraph's counterpart of the result of Haemers and Tonchev by restricting the parameters to $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$ was shown in [4], which says there exists some imprimitive nonsymmetric association scheme if and only if there exists a skew-Bush-type Hadamard matrix. In this section, we show digraph's counterpart of the result of Wallis [12], namely characterize the skew-Bush-type Hadamard matrices in terms of the notion of doubly regular asymmetric digraphs with a similar property to the undirected case.

Proposition 4.1. *The following are equivalent.*

- (i) *There exists a skew-Bush-type Hadamard matrix of order $4n^2$.*

- (ii) *There exists a doubly regular asymmetric digraph with parameters $(4n^2, 2n^2 - n, n^2 - n)$ such that the vertex set is decomposed into $2n$ disjoint cocliques of size $2n$.*

Proof. (i) \Rightarrow (ii): Let H be a skew-Bush-type Hadamard matrix of order $4n^2$. Define a $(0, 1)$ -matrix $A = \frac{1}{2}(J_{4n^2} - H)$. Since $H - I_{2n} \otimes J_{2n}$ is skew-symmetric, A satisfies that $A + A^T = J_{4n^2} - I_{2n} \otimes J_{2n}$. Thus A is the adjacency matrix of a digraph whose vertex set is decomposed into disjoint $2n$ cliques of size $2n$. Since H is a regular Hadamard matrix in particular, it follows that A satisfies the equation $AA^T = n^2 I_{4n^2} + (n^2 - n)J_{4n^2}$. This shows that A is the adjacency matrix of a doubly regular asymmetric digraph with the desired parameters.

(ii) \Rightarrow (i): Let Γ be a doubly regular asymmetric digraph with parameters $(4n^2, 2n^2 - n, n^2 - n)$ with the property that the vertex set is decomposed into $2n$ disjoint cocliques of size $2n$. Let A be the adjacency matrix of Γ . Since Γ is decomposed into $2n$ disjoint cocliques of size $2n$, after a suitable rearranging the ordering of the vertices, we may assume that $A + I_{2n} \otimes J_{2n}$ is a $(0, 1)$ -matrix. Let $H = A - A^T + I_{2n} \otimes J_{2n}$, and set H_{ij}, A_{ij} ($i, j \in \{1, \dots, 2n\}$) to be $2n \times 2n$ matrices such that $H = (H_{ij})_{i,j=1}^{2n}$ and $A = (A_{ij})_{i,j=1}^{2n}$. Then H is a $(1, -1)$ -matrix, and the direct calculation shows that H is a Hadamard matrix. It is clear that each diagonal block of size $2n$ is J_n and $H - I_{2n} \otimes J_{2n}$ is skew-symmetric. By Lemma 2.1 the eigenvalues of A are $2n^2 - n, \pm\sqrt{-1}n, -n$. As is shown in Remark 3.2, the disjoint $2n$ cocliques represented as the main diagonal blocks of A attain the upper bound in Proposition 3.1, and thus by Proposition 3.1(ii) we have $A_{ij}J_{2n} = J_{2n}A_{ij} = nJ_{2n}$ for any distinct i, j , namely $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$. Therefore H is a skew-Bush-type Hadamard matrix. \square

5 Regular biangular matrices and association schemes

In [6] they constructed association schemes from a Hadamard matrix of order n and mutually orthogonal Latin squares of order $n - 1$. In this section, we construct some association scheme from a Hadamard matrix of order n and a single Latin with some properties of order $n - 1$. Some relation graphs of the association schemes have the property that its vertex set is decomposed into disjoint cocliques attaining the bound in Proposition 3.1.

An (α, β) -biangular matrix of order n is an $n \times n$ $(1, -1)$ -matrix H such that the inner products of its normalized rows of H are in $\{\alpha, \beta\}$ [6]. An (α, β) -biangular matrix H of order nm is called *regular* if the rows of H can be partitioned into m -classes of size n each in such a way that:

- (i) $|\langle u, v \rangle| = \alpha$ for each distinct pair u, v in the same class,
- (ii) $|\langle u, v \rangle| = \beta$ for each pair u, v belonging to different classes.

We will use the following lemma proven in [9].

Lemma 5.1. *If there exists a Hadamard matrix, then there exist symmetric $(1, -1)$ -matrices C_1, C_2, \dots, C_n such that:*

- (i) $C_1 = J_n$.
- (ii) $C_i C_j = 0$, $1 \leq i \neq j \leq n$.
- (iii) $C_i^2 = nC_i$, $1 \leq i \leq n$.
- (iv) $\sum_{i=1}^n C_i = nI_n$.

It follows from these conditions that the row sums and column sums are 0 for C_i , $i \neq 1$, and that

$$\sum_{i=2}^n C_i^2 = n^2 I_n - nJ_n.$$

Proof. Letting H be a normalized Hadamard matrix with i -th row h_i for $i \in \{1, \dots, n\}$, set $C_i = h_i^T h_i$. Then C_1, \dots, C_n satisfy the conditions (i)-(iv). \square

Let $H = (H_{ij})_{i,j=1}^n$ be a regular (α, β) -biangular matrix of order nm , where each H_{ij} is a square matrix of order m and the rows in i -th block are in the same class for any $i \in \{1, \dots, m\}$. The regular (α, β) -biangular matrix H is said to be of *skew-symmetric* if $H_{ij}^T = -H_{ji}$ for any distinct $i, j \in \{1, \dots, n\}$.

Theorem 5.2. *Let n be the order of a Hadamard matrix. Then the following hold.*

- (i) *There is a symmetric regular $(0, \frac{1}{n-1})$ -biangular matrix of order $n(n-1)$.*
- (ii) *There is a skew-symmetric regular $(0, \frac{1}{n-1})$ -biangular matrix of order $n(n-1)$.*

Proof. Let H be a normalized Hadamard matrix, and let L be an addition table of \mathbb{Z}_{n-1} . Then L is a symmetric Latin square with (i, j) -entry denoted by $l(i, j)$. We regard L as a Latin square on the set $\{2, \dots, n\}$.

Starting with a symmetric Latin square on the set $\{2, \dots, n\}$ and substituting i with C_i from Lemma 5.1 for $i \in \{2, \dots, n\}$, we obtain a matrix which we will denote by M . Clearly M is a $(1, -1)$ -matrix of order $n(n-1)$. It follows from Lemma 5.1 that MM^T is a block matrix with all diagonal blocks equal to $n^2 I_n - nJ_n$ by Lemma 5.1, and off-diagonal blocks equal to zero matrix by Lemma 5.1 (ii) and the property of L being a Latin square. This completes the proof of (i).

For a construction (ii), define $M = (M_{ij})_{i,j=1}^n$ by $M_{ij} = C_{l(i,j)}$ for $i \leq j$ and $M_{ij} = -C_{l(i,j)}$ for $i > j$. Then it is easy to see that the matrix M is the desired skew-symmetric biangular matrix. \square

More precisely, the matrices M in Theorem 5.2 (i), (ii) satisfy the following equation:

$$MM^T = n(n-1)I_{n(n-1)} - nI_{n-1} \otimes (J_n - I_n). \quad (8)$$

We decompose M into disjoint $(0, 1)$ -matrices A_0, A_1, \dots, A_4 defined as follows:

$$\begin{aligned} M &= A_0 + A_1 - A_2 + A_3 - A_4 \\ A_0 &= I_{n(n-1)} \\ A_1 + A_2 &= (J_{n-1} - I_{n-1}) \otimes J_n, \\ A_0 + A_3 + A_4 &= I_{n-1} \otimes J_n. \end{aligned} \tag{9}$$

$$\tag{10}$$

Note that $A_1 = A_1^T, A_2 = A_2^T$ if M is a symmetric regular biangular matrix and $A_1 = A_2^T$ if M is a skew-symmetric regular biangular matrix, and A_3, A_4 are symmetric in both cases.

Theorem 5.3. (i) *The set of matrices $\{A_0, A_1, A_2, A_3, A_4\}$ forms a symmetric association scheme if M is a symmetric regular biangular matrix.*

(ii) *The set of matrices $\{A_0, A_1, A_2, A_3, A_4\}$ forms a nonsymmetric association scheme if M is a skew-symmetric regular biangular matrix.*

Proof. In both cases, the proof is the same as follows. Let $\mathcal{A} := \text{span}_{\mathbb{R}}\{A_0, A_1, \dots, A_4\}$. Since each block matrix of A_i for any i has a constant row and column sum, we have

$$A_i(I_{n-1} \otimes J_n) = (I_{n-1} \otimes J_n)A_i \in \mathcal{A}, \tag{11}$$

$$A_i((J_{n-1} - I_{n-1}) \otimes J_n) = ((J_{n-1} - I_{n-1}) \otimes J_n)A_i \in \mathcal{A}. \tag{12}$$

First we show $A_i A_j \in \mathcal{A}$ for $i, j \in \{3, 4\}$. By Lemma 5.1 (iii), we have $(A_0 + A_3 - A_4)^2 = n(A_0 + A_3 - A_4)$. By (10) and (11) we have $A_4^2 \in \mathcal{A}$. Similarly $A_i A_j \in \mathcal{A}$ for others $i, j \in \{3, 4\}$.

Next we show $A_i A_j \in \mathcal{A}$ for $i \in \{1, 2\}, j \in \{3, 4\}$ or $i \in \{3, 4\}, j \in \{1, 2\}$. By Lemma 5.1 (ii), we have $(A_1 - A_2)(A_0 + A_3 - A_4) = 0$. By (9), (10), (11) and (12), we have $A_2 A_4 \in \mathcal{A}$. Similarly $A_i A_j \in \mathcal{A}$ for others $i \in \{1, 2\}, j \in \{3, 4\}$ or $i \in \{3, 4\}, j \in \{1, 2\}$.

Finally we show $A_i A_j \in \mathcal{A}$ for $i, j \in \{1, 2\}$. By (8) we have $(A_1 - A_2)^2 \in \mathcal{A}$. By (9) and (12), we have $A_i A_j \in \mathcal{A}$ for $i, j \in \{1, 2\}$. Thus this completes the proof. \square

The first eigenmatrices in Theorem 5.3 (i), (ii) are as follows respectively:

$$P = \begin{pmatrix} 1 & \frac{n(n-2)}{2} & \frac{n(n-2)}{2} & \frac{n-2}{2} & \frac{n}{2} \\ 1 & 0 & 0 & \frac{n-2}{2} & -\frac{n}{2} \\ 1 & -\frac{n}{2} & -\frac{n}{2} & \frac{n-2}{2} & \frac{n}{2} \\ 1 & -\frac{n}{2} & \frac{n}{2} & -1 & 0 \\ 1 & \frac{n}{2} & -\frac{n}{2} & -1 & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & \frac{n(n-2)}{2} & \frac{n(n-2)}{2} & \frac{n-2}{2} & \frac{n}{2} \\ 1 & 0 & 0 & \frac{n-2}{2} & -\frac{n}{2} \\ 1 & -\frac{n}{2} & -\frac{n}{2} & \frac{n-2}{2} & \frac{n}{2} \\ 1 & -\frac{\sqrt{-1}n}{2} & \frac{\sqrt{-1}n}{2} & -1 & 0 \\ 1 & \frac{\sqrt{-1}n}{2} & -\frac{\sqrt{-1}n}{2} & -1 & 0 \end{pmatrix}.$$

See Appendices A, B for the intersection numbers and second eigenmatrices. Consider relation graphs with adjacency matrix A_1, A_2 in both association schemes. As Proposition 3.1 shows, each main diagonal block of A_1, A_2 represents a coclique and A_4 corresponds to a partition of the vertex set by cliques attaining the bound in Proposition 3.1.

6 Twin asymmetric designs and association schemes

Finally we focus on normally regular digraphs with $\lambda = \mu$, or equivalently doubly regular asymmetric graphs. If an incidence matrix N of a symmetric design is such that $N + N^T$ is a $(0, 1)$ -matrix, then N is an adjacency matrix of a doubly regular asymmetric digraph, and vice versa. Our main reference for this section is [7]. We will refer to a doubly regular asymmetric digraph with parameters (v, k, λ) as a $DRAD(v, k, \lambda)$. Symmetric (v, k, λ) -designs $\mathbf{D} = (X, \mathcal{B})$ and $\mathbf{D}' = (X, \mathcal{B}')$ are called *twin designs* if there is a bijection $f: \mathcal{B} \rightarrow \mathcal{B}'$ such that every block $B \in \mathcal{B}$ is disjoint from $f(B)$. In general, it is not easy to find twin symmetric designs. However, if Γ is a $DRAD(v, k, \lambda)$ and Γ' is the digraph obtained by reversing the direction of every arc of Γ , then the corresponding symmetric designs are twins. The following theorem is proven in [7].

Theorem 6.1. *Let h be a positive integer such that there exists a Hadamard matrix of order $2h$. If $p = (2h - 1)^2$ is a prime power, then, for any positive integer d , there exists a*

$$DRAD\left(\frac{h(p^{2d} - 1)}{h + 1}, hp^{2d}, h(h + 1)p^{2d-1}\right). \quad (13)$$

The construction makes use of *skew balanced generalized weighing matrices* and Bush-type Hadamard matrices constructed as in Lemma 5.1 from a Hadamard matrix of order $2h$. We illustrate this by an example which relates to the special case of the theorem which used in this note.

Example 6.2. We start with a BGW(10, 9, 8) over the cyclic group C_8 . Let

$$W = [w_{ij}] = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & 3 & 7 & 5 & 6 & 8 & 1 & 4 & 2 \\ 4 & 7 & 0 & 3 & 8 & 5 & 6 & 2 & 1 & 4 \\ 4 & 3 & 7 & 0 & 6 & 8 & 5 & 4 & 2 & 1 \\ 4 & 1 & 4 & 2 & 0 & 3 & 7 & 5 & 6 & 8 \\ 4 & 2 & 1 & 4 & 7 & 0 & 3 & 8 & 5 & 6 \\ 4 & 4 & 2 & 1 & 3 & 7 & 0 & 6 & 8 & 5 \\ 4 & 5 & 6 & 8 & 1 & 4 & 2 & 0 & 3 & 7 \\ 4 & 8 & 5 & 6 & 2 & 1 & 4 & 7 & 0 & 3 \\ 4 & 6 & 8 & 5 & 4 & 2 & 1 & 3 & 7 & 0 \end{pmatrix}.$$

Then W is a skew BGW(10, 9, 8) over the cyclic group $C_8 = \langle g \rangle$ generated by the matrix

$$g = \begin{pmatrix} 0 & I_4 & 0 & 0 \\ 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & I_4 \\ -I_4 & 0 & 0 & 0 \end{pmatrix},$$

where the number i in G denotes g^i for $i = 1, 2, \dots, 8$. Let

$$H = \begin{pmatrix} 0 & C_2 & C_3 & C_4 \\ -C_4 & 0 & C_2 & C_3 \\ -C_3 & -C_4 & 0 & C_2 \\ -C_2 & -C_3 & -C_4 & 0 \end{pmatrix},$$

where C_2, C_3, C_4 are those constructed in 5.1 from a normalized Hadamard matrix of order 4 and 0 denotes the zero matrix of order 16.

Let

$$R = \begin{pmatrix} 0 & 0 & 0 & I_4 \\ 0 & 0 & I_4 & 0 \\ 0 & I_4 & 0 & 0 \\ I_4 & 0 & 0 & 0 \end{pmatrix}.$$

Let $G = [Hw_{ij}R]$, then G can be splitted to parts, namely the positive and negative part, to form a twin skew symmetric $(160, 54, 18)$ design on 160 vertices.

To do this, keep all the 1-entries in G , change all the -1 -entries to 0 and let A_1 be the $(0, 1)$ -matrix obtained. Then, A_1 is the incidence matrix of a symmetric $(160, 54, 18)$ design. Furthermore, $A_1 + A_1^T$ is a $(0, 1)$ -matrix. So, A_1 is the adjacency matrix of a doubly regular asymmetric digraph. Now change all the 1-entries in G to 0, all -1 -entries to 1 and let A_2 be the $(0, 1)$ -matrix obtained. Then $A_2 = A_1^T$, so A_1 and A_2 are twins. We refer the reader to [7] for the general construction.

We now use the sequence of doubly regular digraphs obtained from the above theorem for $d = 1$ to deduce the existence of some association schemes of class five. The general case corresponding to any positive integer d will appear elsewhere.

Theorem 6.3. *Let $h = 2n$ be a positive integer for which there is a Hadamard matrix of order h and $p = 2n - 1$ is a prime power. Consider the skew $BGW(p^2 + 1, p^2, p^2 - 1)$ over the cyclic group of order $4n$ and the twin design constructed in [7] for $d = 1$.*

Let A_1 be the plus and A_2 the minus twin, $A_4 = I_{2n(p^2+1)} \otimes (J_{2n} - I_{2n})$, $A_5 = I_{p^2+1} \otimes (J_{4n^2} - I_{2n} \otimes J_{2n})$.

Then $\{A_0 = I_{4n^2(p^2+1)}, A_1, A_2, A_3 = J_{4n^2(p^2+1)} - A_1 - A_2 - A_4 - A_5, A_4, A_5\}$ forms a nonsymmetric association scheme of class 5 with the following intersection numbers. Note that $A_1^T = A_2, A_3, A_4, A_5$ are symmetric.

- $A_1 A_1 = A_2 A_2 = (n-1)(2n-1)(2n^2-n)(A_1 + A_2 + A_3 + A_5) + n^2(2n-1)^2 A_4.$
- $A_1 A_2 = n^2(2n-1)^2 A_0 + (2n-1)^2(n^2-n)J.$
- $A_1 A_3 = A_2 A_3 = 2n(n-1)(2n-1)(A_1 + A_2 + A_3) + n(2n-1)^2 A_5.$
- $A_1 A_4 = (n-1)A_1 + nA_2.$

- $A_1A_5 = A_2A_5 = 2n(n-1)(A_1 + A_2) + n(2n-1)A_3$.
- $A_2A_4 = nA_1 + (n-1)A_2$.
- $A_3A_3 = 2n(2n-1)^2A_0 + 4n(n-1)(A_1 + A_2 + A_3) + 2n(2n-1)^2A_4$.
- $A_3A_4 = (2n-1)A_3$.
- $A_3A_5 = 2n(A_1 + A_2)$.
- $A_4A_4 = (2n-1)A_0 + (2n-2)A_4$.
- $A_4A_5 = (2n-1)A_5$.
- $A_5A_5 = 2n(2n-1)A_0 + 2n(2n-1)A_4 + 4n(n-1)A_5$.

Proof. Let $W = [w_{ij}]$ be a skew $BGW(p^2 + 1, p^2, p^2 - 1)$ over a cyclic group of order $4n$ generated by a negacirculant matrix of order $4n$ as described in [7]. Let $R = R_{2n} \otimes I_{2n}$, where R_{2n} denotes the back identity matrix of order $2n$ and I_{2n} is the identity matrix of order $2n$. Then $A_3 = [|w_{ij}|R]$. The identities for $A_1A_2 = A_2A_1$ follows from the fact that each of A_1 and A_2 are the incidence matrices of a symmetric $(p^2 + 1)4n^2, p^2(2n^2 - n), p^2(n^2 - n)$ designs and $A_1^T = A_2$. The numbers for A_1A_1 and A_2A_2 follows from the fact that the symmetric matrix $A_1 + A_2$ has a simple structure and we make use of it in finding the numbers for other products involving A_1 and A_2 . The relation related to A_3 follows from the observation that $A_3 = [|w_{ij}|R]$ and $A_1 + A_2 + A_3 = J_{4n^2(p^2+1)} - I_{p^2+1} \otimes J_{4n^2}$. The remaining numbers are not hard to calculate. \square

The eigenmatrices P, Q are given as follows:

$$P = \begin{pmatrix} 1 & n(2n-1)^3 & n(2n-1)^3 & 2n(2n-1)^2 & 2n-1 & 2n(2n-1) \\ 1 & n(2n-1) & n(2n-1) & -2n(2n-1) & 2n-1 & -2n \\ 1 & n(2n-1)\sqrt{-1} & -n(2n-1)\sqrt{-1} & 0 & -1 & 0 \\ 1 & -n(2n-1)\sqrt{-1} & n(2n-1)\sqrt{-1} & 0 & -1 & 0 \\ 1 & -n(2n-1) & -n(2n-1) & -2n & 2n-1 & 2n(2n-1) \\ 1 & -n(2n-1) & -n(2n-1) & -2n(2n-1) & 2n-1 & -2n \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & (2n-1)m & 2n(2n-1)m & 2n(2n-1)m & (2n-1)^2 & (2n-1)m \\ 1 & \frac{m}{2n-1} & -\frac{2nm\sqrt{-1}}{2n-1} & \frac{2nm\sqrt{-1}}{2n-1} & -1 & -\frac{m}{2n-1} \\ 1 & \frac{m}{2n-1} & \frac{2nm\sqrt{-1}}{2n-1} & -\frac{2nm\sqrt{-1}}{2n-1} & -1 & -\frac{m}{2n-1} \\ 1 & -m & 0 & 0 & -1 & -2n+1 \\ 1 & m & -2nm & -2nm & (2n-1)^2 & m \\ 1 & -m & 0 & 0 & (2n-1)^2 & -m \end{pmatrix},$$

where $m = 2n^2 - 2n + 1$. By the definition of A_4, A_5 , we have $A_4 + A_5 = I_{p^2+1} \otimes (J_{4n^2} - I_{4n^2})$. The cocliques of the digraphs whose adjacency matrices are A_1, A_2 corresponding to the main diagonal blocks of $A_4 + A_5$ attain the upper bound in Proposition 3.1.

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Appendix A Parameters of the association scheme in Theorem 5.3(i)

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{n^2-2n}{2} & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2-4n}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n}{4} - 1 & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \end{pmatrix} \\
B_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2}{4} & \frac{n^2-2n}{4} \\ \frac{n^2-2n}{2} & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2-4n}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n}{4} & \frac{n-4}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \end{pmatrix} \\
B_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{n}{4} - 1 & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} - 1 & 0 & 0 \\ \frac{n}{2} - 1 & 0 & 0 & \frac{n}{2} - 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2} - 1 \end{pmatrix} \\
B_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2} - 1 \\ \frac{n}{2} & 0 & 0 & \frac{n}{2} & 0 \end{pmatrix} \\
Q &= \begin{pmatrix} 1 & n-1 & n-2 & \frac{(n-1)(n-2)}{2} & \frac{(n-1)(n-2)}{2} \\ 1 & 0 & -1 & -\frac{n-1}{2} & \frac{n-1}{2} \\ 1 & 0 & -1 & \frac{n-1}{2} & -\frac{n-1}{2} \\ 1 & n-1 & n-2 & -n+1 & -n+1 \\ 1 & -n+1 & n-2 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Appendix B Parameters of the association scheme in Theorem 5.3(ii)

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2}{4} & \frac{n^2-2n}{4} \\ \frac{n^2-2n}{2} & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2-4n}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n}{4} - 1 & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \end{pmatrix} \\
B_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{n^2-2n}{2} & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2-4n}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n^2-3n}{4} & \frac{n^2-3n}{4} & \frac{n^2}{4} & \frac{n^2-2n}{4} \\ 0 & \frac{n}{4} & \frac{n-4}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \end{pmatrix} \\
B_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{n}{4} - 1 & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} - 1 & 0 & 0 \\ \frac{n}{2} - 1 & 0 & 0 & \frac{n}{2} - 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2} - 1 \end{pmatrix} \\
B_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{4} & \frac{n}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{n}{2} - 1 \\ \frac{n}{2} & 0 & 0 & \frac{n}{2} & 0 \end{pmatrix} \\
Q &= \begin{pmatrix} 1 & n-1 & n-2 & \frac{(n-1)(n-2)}{2} & \frac{(n-1)(n-2)}{2} \\ 1 & 0 & -1 & -\frac{\sqrt{-1}(n-1)}{2} & \frac{\sqrt{-1}(n-1)}{2} \\ 1 & 0 & -1 & \frac{\sqrt{-1}(n-1)}{2} & -\frac{\sqrt{-1}(n-1)}{2} \\ 1 & n-1 & n-2 & -n+1 & -n+1 \\ 1 & -n+1 & n-2 & 0 & 0 \end{pmatrix}
\end{aligned}$$